Spin waves and density of states in disordered ferromagnetic substances

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A study is made of spin waves and the density of states at low frequencies and $T \ll T_c$ in binary alloys A_cB_{1-c} with a small concentration of magnetic atoms $c \ll 1$. It is assumed that the exchange-interaction energy falls off exponentially with distance between the magnetic atoms. It is demonstrated that weakly damped spin waves with a quadratic dispersion law can be propagated in such a system, the mass of the waves being inversely proportional to c when the radius R of the exchange energy exceeds the mean distance r_c between impurities and diminishes exponentially with increasing c when $R \ll r_c$. The attenuation of the spin waves is proportional to $q^2\omega^{3/2}$ and is slight for long waves. The density of states is determined mainly not by the actual spin waves but by a broadening of the spin levels in the molecular field, a broadening associated with virtual, noncoherent absorption and emission of spin waves. Owing to the local nature of the density states, the appropriate contribution to electrical resistivity is proportional to $T^{3/2}$. For the case $R \gg r_c$, a concentration dependence has been found for the density of states and for the temperature-dependent corrections to the mean spin and the electrical resistivity.

1. INTRODUCTION

At the present time, intensive studies have been made of the properties of binary alloys of magnetic substances with nonmagnetic ones, in which the magnetic atoms are disposed in random fashion (see, e.g.,[1-3]). In all these studies, only exchange interaction of the nearest neighbors in the lattice is taken into account. Ferromagnetism exists in this case if the concentration of the magnetic atoms is not too small (on the order of 50%). At the same time, we know of a large number of strongly diluted ferromagnetic alloys (Pd with Fe, Co, or Mn impurities, Ni_3Ga with Fe impurity, etc.), which retain their ferromagnetic properties at magnetic-atom concentrations smaller than 0.1%[4-7].

In the present paper we consider the binary alloy A_cB_{1-c}, in which the concentration of the magnetic atoms is $c \ll 1$. The exchange interaction of these atoms is described by the Heisenberg Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j}\varphi_i\varphi_j V(\mathbf{r}_i - \mathbf{r}_j)\mathbf{S}_i\mathbf{S}_j. \qquad (1)$$

Here S_i is the spin localized at the site of the lattice r_i. The quantities φ are equal to unity at sites occupied with magnetic atoms and to zero in the remaining sites.

We assume that the exchange-interaction energy decreases exponentially with distance:

$$V(r) = V_0 e^{-r/R}. \qquad (2)$$

The properties of such a ferromagnet depend strongly on the ratio of the radius R of the potential to the average distance between impurities $r_c \approx n^{-1/3}$, where n is the number of magnetic atoms per unit volume. If $R \gg r_c$, then each spin interacts effectively with a large number of other spins located in a volume of radius R. Their number is of the order of $\nu = \frac{4}{3}\pi n R^3 \gg 1$. Therefore, in spite of the strong fluctuations in the arrangement of the spins, the energy of the exchange interaction of each of them depends little on the particular configuration. Such a situation is realized apparently in PdNi alloys at Ni concentrations $c > 3\%$, in which the radius is $R \approx 7$ Å[8]. In the alloys PdFe and PdCo, at Fe and Co magnetic-atom concentrations larger than 2%, the magnetization of the system is close to homogeneous[9], i.e., the spatial fluctuations of the molecular field that is exerted on the localized spins of Fe and Co by the d electrons of Pd are small. In this respect, such alloys are similar to a system of localized spins with $R \gg r_c$.

If $R \ll r_c$, i.e., $\nu \ll 1$, then most spins interact effectively only with one nearest spin. Although for most magnetic atoms the nearest magnetic neighbor is located at a distance on the order of r_c, the energy of exchange interaction with it, which varies significantly over distances on the order of R, depends strongly on the particular configuration. This situation is realized in dilute alloys of the PdFe type at an iron concentration $c < 1\%$[9,10].

In the case $\nu \gg 1$ (large spin density), it is possible to develop, with the aid of the Edwards and Jones diagram technique[2], a perturbation theory for the Green's function of the transverse components of the spin operator. The small parameter of the perturbation theory is ν^{-1}, i.e., we use the smallness of the spatial fluctuations of the molecular-field energy. In such a system, there exist weakly-damped magnons whose wavelength exceeds the average distance between the atoms. Their existence is a consequence of the total-spin conservation law. The magnon spectrum is

$$\omega_q = Sc[V(0) - V(q)],$$

where V(q) is the Fourier transform of the potential V(r). The relative damping of the magnons is of the order of $(qr_c)^3$. It turns out that at $T \ll T_c$ the real magnons do not determine the state density $\rho(\omega)$ for low frequencies. The main contribution to $\rho(\omega)$, proportional to $\sqrt{\omega}$, is made by the broadening of the levels of the spins in the molecular field, a broadening due to the incoherent emission (absorption) of virtual magnons. For this reason, real magnons appear neither in the thermodynamic quantities nor in the electric resistivity, which is due mainly to local elastic and inelastic scattering of the electrons and is proportional to $T^{3/2}$. It is possible to observe real magnons by investigating the small-angle scattering of the neutrons.

In the case $\nu \ll 1$ (low spin density) the situation is much more complicated. Owing to the strong fluctuations of the molecular field, it is impossible to develop a consistent perturbation theory for the Green's function. Nonetheless, our analysis leads to the conclusion that owing to the spin-conservation law there exist in this case, too, long-wave magnons with the quadratic dispersion law. Their mass, as shown by an estimate, is proportional to $\nu^{-5/12}\exp(2/3\sqrt{3}\nu)$, i.e., the mass decreases linearly with increasing ν.

The state density connected with the real magnons and with the broadening of the local levels of the spins that are located at distances on the order of the mean distances between one another, is proportional to $\sqrt{\omega}$ just as in the case $\nu \gg 1$. At the same time, in a system with a low magnetic-atom density there is one more contribution to the density of states, connected with the local flip of the spins whose distances from the neighbors are greater than the average distance[10]. As shown earlier[10], at sufficiently small ν and not too low energies, this is the main contribution.

2. LARGE DENSITY OF MAGNETIC ATOMS

A binary alloy with a large density of magnetic atoms ($\nu \gg 1$) is, in the case $c \ll 1$, an example of a ferromagnet with strong fluctuations in the arrangement of the magnetic centers, for which it is possible to develop a consistent perturbation theory in the parameter ν^{-1} for the Green's function of the transverse components of the spin operator.

We shall use for $T = 0$ a diagram technique analogous to that of Edwards and Jones[2]. We introduce a Green's function for the transverse components of the spin

$$2SG_{ll'}(t) = -i\langle TS_l^-(t)S_{l'}^+(0)\varphi_l\varphi_{l'}\rangle, \tag{3}$$

where $S_l^\pm = S_x \pm iS_y$. Its temporal Fourier transform is

$$G_{ll'}(\omega) = \int e^{i\omega t}G_{ll'}(t)\,dt$$

and is determined by the equation[2]

$$\sum_m \left\{\omega\delta_{lm} - S\left[\delta_{lm}\sum_i V_{li}\varphi_i - \varphi_l V_{lm}\right]\right\} G_{ml'}(\omega) = \delta_{ll'}\varphi_l. \tag{4}$$

Averaging over the distribution of the magnetic atoms in the lattice is best carried out with the aid of the distribution function

$$F(\varphi) = c\delta(\varphi - 1) + (1 - c)\delta(\varphi).$$

After averaging over the arrangement of the magnetic atoms, the Green's function, naturally depends only on the coordinate difference $r_{l-l'}$. Its Fourier transform is

$$G(\mathbf{q}, \omega) = \sum_l \exp(-i\mathbf{q}\mathbf{r}_l)G(\mathbf{r}_l).$$

As the zeroth approximation of the perturbation theory, we use the solution of an equation obtained from (4) by averaging the coefficients over the locations of the impurities,

$$G_0(q, \omega) = c\,/\,(\omega - \omega_q), \tag{5}$$

$$\omega_q = Sc[V(0) - V(\mathbf{q})], \tag{6}$$

$$V(\mathbf{q}) = \sum_l V(r_l)\exp(-i\mathbf{q}\mathbf{r}_l).$$

At small $q \ll R^{-1}$ we have

$$\omega_q = q^2\,/\,2m_s. \tag{7}$$

The reciprocal magnon mass, according to (2), is equal to

$$m_s^{-1} = 24S\nu V_0 R^2. \tag{8}$$

The rules formulated by Edwards and Jones[2] for constructing the diagrams can be easily reduced to the following. The diagrams of (1) consist of the following: 1) Solid horizontal lines, to which the function $c^{-1}G_0(q, \omega)$ is set in correspondence. 2) Wavy lines, corresponding to the function $S[V(q - p) - V(p)]$. The corresponding vertex is shown in Fig. 1a. 3) Points from which n lines emerge; they correspond to cumulants that represent polynomials $P_n(c)$ of n-th degree[11]. In particular,

$$P_1 = \langle\varphi\rangle, \quad P_2 = \langle\varphi^2\rangle - \langle\varphi\rangle^2.$$

At small c we have $P_n(c) = c$ for all the cumulants. 4) On the right end of the solid line there is always a point from which wavy lines can emerge. This point can be set in correspondence with a cumulant (P_1 or P_2 in Fig. 1).

The aggregate of the diagrams of Fig. 1b leads in the usual manner to the self-energy part $\Sigma(q, \omega)$. The aggregate of diagrams of type 1c leads to a renormalization of the residue of the Green's function. The Green's function takes the form

$$G(q, \omega) = c[1 + L(q, \omega)]\,/\,[\omega - \omega_q - \Sigma(q, \omega)], \tag{9}$$

where $L(q, \omega)$ is the sum of all the irreducible diagrams in which a wavy line emerges from the extreme right point on the thick line. The simplest diagram that contributes to $L(q, \omega)$ is 1c without the left-hand free line.

The contribution made to Σ and L by the simplest diagrams 1b and 1c is equal to

$$\Sigma_1(q, \omega) = S^2c(1 - c)\Omega_0\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{[V(\mathbf{q}-\mathbf{k}) - V(\mathbf{k})][V(\mathbf{q}-\mathbf{k}) - V(\mathbf{q})]}{\omega - \omega_k + i\delta}, \tag{10}$$

$$L_1(q, \omega) = S(1 - c)\Omega_0\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{V(\mathbf{q}-\mathbf{k}) - V(\mathbf{k})}{\omega - \omega_k + i\delta}, \tag{11}$$

where Ω_0 is the volume of the unit cell. From (10) and (11) we see that as $q \to 0$ the quantities Σ_1 and L_1 also tend to zero. This is an important property possessed by all diagrams. It follows mathematically from the fact that the wavy line is set in correspondence with the difference $V(q - p) - V(p)$, which vanishes as $q \to 0$. Physically this is connected with the fact that the initial Hamiltonian (1) commutes with the total angular momentum, i.e., the spin is conserved.

The Fourier transform of the potential (2) is

$$V(\mathbf{k}) = 8\pi V_0\,/\,R\Omega_0(k^2 + R^{-2})^2, \tag{12}$$

i.e., at $k \gg R^{-1}$ the quantity $V(k)$ decreases rapidly. Therefore in integrals of the type (10) and (11) the phase volume is of the order of $R^{-3} \ll \Omega_0^{-1}$. Consequently,

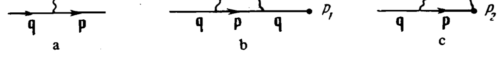

FIG. 1.

each integration with respect to the internal momenta introduces a smallness of the order of Ω_0/R^3. A similar property of the diagrams was used in the construction of a perturbation theory in the papers of Vaks, Larkin, and Pikin[12]. We are interested in frequencies $\omega \ll cV(0)$. Then the important role in (10) and (11) is played by $\omega_k \gtrsim \omega$. Since $\omega_k \sim c$, it follows that the denominator in (10) and (11) is proportional to c. Thus,

$$L_1 \approx \Omega_0 / R^3 c \approx \nu^{-1} \ll 1.$$

It is easy to see that in more complicated diagrams each integration introduces an additional smallness of order ν^{-1}. To calculate Σ and L it therefore suffices to confine oneself to expressions (10) and (11). It will be shown later on that this conclusion is valid in the case of low frequencies $\omega \ll cV(0)$ for all momenta, up to $q \approx \Omega_0^{-1/3}$. It follows from (10) that $\operatorname{Re}\Sigma \sim q^2$ at small q, i.e., Re Σ leads to a small renormalization (of order ν^{-1}) of the magnon mass. Thus, the spin-wave spectrum is described by formulas (6), (7), and (8)[1)].

For the damping of the spin waves we obtain from (9) and (10)

$$\gamma(q,\omega) = \pi S^2 \Omega_0 c(1-c) \int \frac{d^3k}{(2\pi)^3} \delta(\omega - \omega_k) \times [V(\mathbf{q}-\mathbf{k}) - V(\mathbf{k})][V(\mathbf{q}-\mathbf{k}) - V(\mathbf{q})]. \tag{13}$$

At small ω and **q**, expanding V(**q** - **k**) in powers of **q** and **k**, we obtain

$$\gamma(q,\omega) = \frac{7\sqrt{2}}{24\pi} \Omega_0 \frac{(1-c)}{c} \sqrt{m_s}\, q^2 \omega^{3/2}, \tag{14}$$

i.e., the relative damping is

$$\frac{\gamma(q,\omega)}{\omega_q} = \frac{7}{24\pi} \frac{1-c}{c} \Omega_0 q^3 \approx \nu^{-1}(qR)^3 \approx (qr_c)^3 \ll 1. \tag{15}$$

It is easily seen that more complicated diagrams also make a contribution proportional to $q^2\omega^{3/2}$ to the damping. Let us examine, e.g., the diagram of Fig. 2. It corresponds to the following contribution to Σ(q, ω):

$$\Sigma_2(q,\omega) = S^4 \Omega_0^2 c^2 (1-c)^2 \int \frac{d^3\mathbf{k}\, d^3\mathbf{k}_1}{(2\pi)^6} \times \frac{[V(\mathbf{q}-\mathbf{k}) - V(\mathbf{k})][V(\mathbf{k}-\mathbf{k}_1) - V(\mathbf{k}_1)][V(\mathbf{q}-\mathbf{k}) - V(\mathbf{q}-\mathrm{k}+\mathbf{k}_1)]}{(\omega - \omega_k + i\delta)(\omega - \omega_{k_1} + i\delta)} \times \frac{[V(\mathbf{k}-\mathbf{k}_1) - V(\mathbf{k})]}{\omega - \omega_{\mathbf{q}-\mathbf{k}+\mathbf{k}_1} + i\delta}. \tag{16}$$

Im Σ_2 contains $\delta(\omega - \omega_p)$, where p is some internal momentum. Consequently, one of the internal momenta is proportional to $\sqrt{\omega}$. Therefore Im Σ, like Im Σ_1, is proportional to $q^2\omega^{3/2}$. Integration with respect to the other momentum in (16) yields, as already noted, an additional smallness of order ν^{-1}. It is clear that the expressions for Im Σ will have a similar structure in all orders of perturbation theory.

Since the damping is proportional to $q^2\omega^{3/2}$ in all orders in ν^{-1}, expression (4) suffices for the calculation of this damping. Thus, at $c \ll 1$ and $\nu \gg 1$, i.e., in a system of randomly-arranged magnetic moments, in which the energy of exchange interaction with the environment fluctuates little, there exist well-defined magnons of wavelength larger than the average distance between the impurities. At $1 - c \ll 1$, formula (14) describes the damping of long-wave magnons by defects in an almost-ideal ferromagnet. This result was obtained earlier by Callaway[13] and by Edwards and Jones[2].

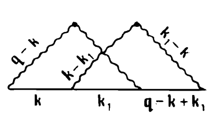

FIG. 2.

The quantity L(q, ω) in (9) does not influence the spin-wave spectrum, but is important for determination of the state density of the magnetic excitations. The real part of L(q, ω) is small in comparison with unity and can be discarded. For Im L at low frequencies we obtain from (11)

$$\operatorname{Im} L_1(q,\omega) = \begin{cases} \pi(1-c)c^{-1}\omega_q \rho_0(\omega), & q \ll R^{-1} \\ S\pi(1-c)c^{-1}V(0)\rho_0(\omega) & q \gg R^{-1}; \end{cases} \tag{17}$$

$$\rho_0(\omega) = \Omega_0 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(\omega - \omega_k). \tag{18}$$

We introduce the density of states

$$\rho(\omega) = -\frac{\Omega_0}{\pi c} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \operatorname{Im} G(k,\omega).$$

As seen from the definition of the Green's function (3), ρ(ω) is normalized to unity. According to (9) we have

$$\rho(\omega) = -\frac{\Omega_0}{\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \operatorname{Im} \frac{1 + L_1(k\ \omega)}{\omega - \omega_k + i\delta}. \tag{19}$$

At $T \ll T_C$, interest attaches to the density of states at small ω. It is seen from (19) that the contribution to ρ(ω) consists of two parts: of the contribution of the pole, i.e., the real spin waves, equal to $\rho_1(\omega) = \rho_0(\omega)$, and a contribution proportional to Im L(k, ω), equal to

$$\rho_2(\omega) = (1-c)c^{-1}\rho_0(\omega).$$

In the calculation of $\rho_2(\omega)$, an important role is played in the integral by large momenta $k \sim a^{-1}$ (a is the lattice constant), so that the denominator can be replaced by -ScV(0). Thus,

$$\rho(\omega) = \rho_1(\omega) + \rho_2(\omega) = \frac{1}{c}\rho_0(\omega) = \frac{\Omega_0 m_s^{3/2} \omega^{1/2}}{\sqrt{2}\pi^2 c}. \tag{20}$$

We emphasize that at $c \ll 1$ we have the inequality $\rho_2(\omega) \gg \rho_1(\omega)$. In other words, although at small ω at q the spin waves are weakly-damped excitations, it is not they which determine the density of states at $\omega \ll cV(0)$. The principal contribution to the density of states is made by the non-pole part of the Green's function at large momenta $q \sim a^{-1}$. This means that the density $\rho_2(\omega)$ is connected with the broadening of the levels of the spins of the magnetic atoms in the molecular field. The broadening is due to the coherent virtual emission (absorption) of the spin waves. As will be shown later on, the local origin of $\rho_2(\omega)$ exerts a strong influence on the temperature dependence of the electric resistivity of dilute ferromagnetic alloys.

3. LOW DENSITY OF MAGNETIC ATOMS

The perturbation theory developed in the preceding section was based on the parameter ν^{-1}. Therefore in the case of low magnetic-atom density, when $\nu \ll 1$, perturbation theory does not make it possible to obtain a closed expression for the Green's function. It is possible, however, to draw qualitative conclusions concerning the structure of the magnetic-excitation spectrum.

As already noted, Re Σ(q, ω) in all orders in ν^{-1} is proportional to q^2 at small q and ω, whereas Im Σ(q, ω) ~ $q^2\omega^{3/2}$. We can therefore expect at $\nu \ll 1$, just as at $\nu \gg 1$,

$$\operatorname{Im}\Sigma(q,\omega) \ll \operatorname{Re}\Sigma(q,\omega),$$

i.e., in a rarefied system of localized spins there can propagate spin waves of large wavelength, the frequency of which is proportional to q^2. Let us estimate the mass of the spin waves.

We represent $\Sigma(q)$ in the form $\Sigma(q) = \Sigma_0 q^2 R^2$. The factor q^2 is the result of the fact that the Fourier components of the interaction of the longitudinal and transverse components of the spins cancel each other with accuracy to q^2R^2. Since this cancellation has already been obtained, it follows that for an order-of-magnitude estimate of the coefficient Σ_0 we can retain in it the interaction of only the longitudinal component of the spins. Let us consider, e.g., expression (10) for Σ_1. Expanding $V(\mathbf{q}-\mathbf{k})$ in powers of q in one factor of the numerator and discarding in the other factor the quantity $V(\mathbf{q})$, which is connected with the interaction of the transverse components, we can write for the numerator the estimate

$$q^2\frac{\partial^2 V(\mathbf{k})}{\partial k^2}V(\mathbf{k}-\mathbf{q}) \approx q^2R^2V(\mathbf{k})V(\mathbf{q}-\mathbf{k}) \approx q^2R^2V^2(\mathbf{k}-\mathbf{q}),$$

i.e., the coefficient at $(qR)^2$ in Σ_1 can be obtained in order of magnitude from diagram 1b, by setting the wavy line in correspondence with $v(\mathbf{q}-\mathbf{k})$.

Thus, if the wavey lines in all diagrams for Σ are set in correspondence not with $V(\mathbf{q}-\mathbf{p}) - V(\mathbf{p})$, but only with $V(\mathbf{q}-\mathbf{p})$, we obtain diagrams for Σ_0. On the other hand, the aggregate of these diagrams gives the self-energy part of the Green's function, calculated from (4) without allowance for the interaction of the transverse components of the spins, i.e., the function

$$\overline{(\omega - H_i)^{-1}},$$

where H_i is the energy of the molecular field acting at the point R_i, and the superior bar denotes averaging over the arrangement of the impurities. Consequently, putting $\Sigma_1 = \Sigma_0 + cV(0)$, we can write

$$(\omega - \Sigma_1)^{-1} = \overline{(\omega - H_i)^{-1}}$$

and at small ω we have

$$\Sigma_1^{-1} = \int_0^\infty \frac{dH}{H}W(H), \qquad (21)$$

where $W(H)$ is the distribution function of the molecular field[10]. It is shown in the Appendix that the integral (21) is of the order of $\exp(2/3\sqrt{3\nu})$. Thus, apart from the pre-exponential factors, we have for the magnon mass $m_s \sim \exp(2/3\sqrt{3\nu})$.

We have seen in the preceding section that $\operatorname{Im} L(\omega)$ is proportional to $\sqrt{\omega}$ in all orders of perturbation theory. Although at $\nu \ll 1$ the perturbation theory series for $\operatorname{Im} L(\omega)$ cannot be summed, one should expect also in the case of small ν to obtain $\operatorname{Im} L(\omega) \sim \sqrt{\omega}$.

We now consider a different method of estimating the magnon mass, which will enable us to obtain also the pre-exponential dependence on the concentration. We rewrite (4) in the form

$$G_{ll'} = \frac{\delta_{ll'}\varphi_l}{\omega - H_l} - \sum_m \frac{SV(l-m)\varphi_l}{\omega - H_l}G_{ml'}, \qquad (22)$$

$$H_l = S\sum_m V(l-m)\varphi_m. \qquad (23)$$

Iterating (22), we obtain

$$G_{ll'} = \frac{\delta_{ll'}\varphi_l}{\omega - H_l} - \frac{SV(l-l')\varphi_l\varphi_{l'}}{(\omega - H_l)(\omega - H_{l'})} + S^2\sum_m \frac{V(l-m)V(m-l')\varphi_l\varphi_m\varphi_{l'}}{(\omega - H_l)(\omega - H_m)(\omega - H_{l'})} + \ldots.$$

We average $G_{ll'}$ over the arrangement of the magnetic impurities at a fixed difference $\mathbf{r}_l - \mathbf{r}_{l'}$. For the Fourier transform of the obtained function of the difference $\mathbf{r}_l - \mathbf{r}_{l'}$, we obtain

$$\begin{aligned} G(q) = {} & \frac{1}{N}\sum_i \frac{\varphi_l}{\omega - H_l} - \frac{S}{N}\sum_{l,l'} \frac{V(l-l')\varphi_l\varphi_{l'}\exp(-i\mathbf{q}(\mathbf{r}_l - \mathbf{r}_{l'}))}{(\omega - H_l)(\omega - H_{l'})} \\ & + \frac{S^2}{N}\sum_{l,m,l'} \frac{V(l-m)V(m-l')\varphi_l\varphi_m\varphi_{l'}}{(\omega - H_l)(\omega - H_m)(\omega - H_{l'})}\exp(-i\mathbf{q}(\mathbf{r}_l - \mathbf{r}_{l'})) + \ldots. \end{aligned} \qquad (24)$$

Expression (24) contains the ratios of the exchange energy to the spin energy in the molecular field, which fluctuate much less than each of these quantities. Therein lies the advantage of writing down the expression for $G_{ll'}(\omega)$ in the form (22).

To estimate $G(q)$, we can make use of the fact that the molecular field at an impurity located at a distance on the order of r_c from the neighbors is produced only by one nearest neighbor. Indeed, if the distance from this impurity to the nearest neighbor is r, then the probability that it is acted upon by an impurity other than the nearest neighbor is of the order of $r^2Rn \approx \nu(r/R)^2$. This probability is small at $r < r_1 = r_c\nu^{-1/6}$. It is precisely these impurities, which are separated from the nearest neighbor by a distance satisfying this inequality, which will be of importance to us in the sequel.

Let us examine the second term of the series (24). Let $q = 0$. At fixed l, the main contribution to the sum over l' is made by one impurity, namely the one for which l is the nearest neighbor. Then

$$H_l \approx SV(l-l'), \quad V(l-l')/(\omega - H_{l'}) \approx 1.$$

If l is an impurity that is not the nearest neighbor of any other impurity, then it makes no contribution to the sum over l. Consequently, for impurities making an effective contribution to the sum over l and l', the ratio $V(l-l')/(\omega - H_l)$ is practically independent of l, and as a first approximation we can average this quantity over l. At $qr_c \ll 1$ the situation is analogous. Thus, the second term in (24) breaks up into a product of two factors, $A_1(\omega)$ and $B_1(\omega, q)$, where

$$A_1(\omega) = \frac{1}{N}\sum_l \frac{\varphi_l}{\omega - H_l}, \qquad (25)$$

$$B_1(\omega, q) = \frac{S}{Nc}\sum_{l,l'} \frac{V(l-l')\varphi_l\varphi_{l'}\exp(-i\mathbf{q}(\mathbf{r}_l - \mathbf{r}_{l'}))}{\omega - H_l}. \qquad (26)$$

Reasoning in the same manner, we can represent the third term in (24) in the form $A_1(\omega)$, $B_1^2(\omega, q)$, etc. Consequently, for $G^0(\omega, q)$ we obtain in the zeroth approximation

$$G^0(\omega, q) = A_1(\omega)/[1 + B_1(\omega, q)]. \qquad (27)$$

As already noted, from the fact that the total spin is conserved it follows that $G(q = 0) = c/\omega$. We shall show that the function $G^0(q, \omega)$, defined by (27), (25), and (26), satisfies this equation. Using (23), we get

$$1 + B_1(\omega, 0) = 1 + \frac{S}{N}\sum_{l,l'} \frac{V(l-l')\varphi_l\varphi_{l'}}{\omega - H_l} = 1 + \frac{1}{Nc}\sum_l \frac{\varphi_l H_l}{\omega - H_l} = \frac{\omega}{c}A_1(\omega),$$

i.e.,

$$G^0(q=0) = c/\omega.$$

At low frequencies $\omega \ll H_l$ we have

$$G^0(\omega,q) = c\left\{\omega - (\overline{H^{-1}})^{-1}\frac{1}{Nc}\sum_l \frac{\varphi_l[H_l - H_l(q)]}{H_l}\right\}^{-1}, \tag{28}$$

$$H_l(q) = S\sum_{l'} V(l-l')\varphi_{l'}\exp[-i\mathbf{q}(\mathbf{r}_l - \mathbf{r}_{l'})]. \tag{29}$$

In the case $\nu \gg 1$, the fluctuations of the molecular field are small. Then $H_l \approx \overline{H} \sim (\overline{H^{-1}})^{-1}$, and from (28) and (29) we obtain formula (6) for the spin-wave spectrum. If $\nu \ll 1$, then the main contribution to the sum over l and l' in the spectrum is made by $r_{ll'} \approx r_c$. Therefore at $qr_c \ll 1$ the exponential in $H_l(q)$ can be expanded in a series. Since the system of impurities is on the average symmetrical with respect to inversion of the coordinates, the terms that are odd in q drops out. We thus obtain a quadratic spectrum of spin waves, with mass

$$m_s^{-1} = \frac{S}{3}\frac{1}{\overline{H^{-1}}}\frac{1}{Nc}\sum_{l,l'}\frac{V_{l-l'}\varphi_l\varphi_{l'}}{H_l}r_{ll'}^2. \tag{30}$$

In order of magnitude we have

$$m_s \approx \overline{H^{-1}}r_c^{-2}.$$

In calculating $G^0(q)$ we have averaged in (24) over m the quantities of the type

$$V(m-l)\varphi_l/(\omega - H_l).$$

The next approximation are quantities, averaged over m, of the type

$$\sum_n \frac{V(m-n)V(n-l)\varphi_n\varphi_l}{(\omega-H_n)(\omega-H_l)}.$$

The first two terms of the series (24) are taken into account exactly. We obtain

$$\begin{gathered}G^1(\omega, q) = [A_1(\omega) - cA_2(q,\omega)]/[1 - B_2(\omega,q)];\\ A_2(q,\omega) = \frac{S}{Nc}\sum_{l,l'}\frac{V(l-l')\varphi_l\varphi_{l'}}{(\omega-H_l)(\omega-H_{l'})}\exp(-i\mathbf{q}\mathbf{r}_{ll'}),\end{gathered} \tag{31}$$

$$B_2(\omega,q) = \frac{S^2}{Nc}\sum_{l,l'}\frac{V(l-m)V(m-l')\varphi_l\varphi_m\varphi_{l'}}{(\omega-H_m)(\omega-H_{l'})}\exp(-i\mathbf{q}\mathbf{r}_{ll'}). \tag{32}$$

Just as in the preceding case, we can show that $G^1(q=0) = c/\omega$.

At q different from zero and at low frequencies, we obtained from (31) and (32), for the spin-wave spectrum,

$$\omega_q^{(1)} = [B_2(0,q) - B_2(0,0)]/[A_1(0) - A_2(0,0)];$$

$$B_2(0,q) - B_2(0,0) = \frac{1}{Nc}\sum_{l'}\varphi_{l'}\frac{H_{l'}(q) - H_{l'}}{H_l}$$
$$+\frac{S}{Nc}\sum_{m,l'}\frac{\varphi_m\varphi_{l'}}{H_m H_{l'}}V(m-l')[H_m(q) - H_m]\exp(-i\mathbf{q}\mathbf{r}_{ml'}).$$

In the case of a large density of the magnetic atoms, $\omega_q^{(1)}$ differs from $\omega_q^{(0)}$ by a small quantity of order ν^{-1}. For $\nu \ll 1$ and $qr_c \ll 1$ we have

$$A_2(0,0) = \frac{S}{Nc}\sum_{l,l'}\frac{V(l-l')\varphi_l\varphi_{l'}}{H_l H_{l'}} \approx \overline{H^{-1}} = -A_1(0),$$

$$\frac{1}{Nc}\sum_{m,l'}\varphi_m\varphi_{l'}\frac{H_m(q) - H_m}{H_m}\frac{V(m-l')\exp(i\mathbf{q}\mathbf{r}_{ml'})}{H_{l'}} \approx (qr_c)^2,$$

i.e.,

$$\omega_q^{(1)} \approx (\overline{H^{-1}})^{-1}r_c^2 q^2 \approx \omega_q^{(0)}.$$

Quantities containing an increasing number of products of terms of the type $V(m-n)(\omega - H_n)^{-1}$ are averaged out in the next higher approximations, and an ever increasing number of terms of the series (24) is accounted for exactly. It turns out here that the magnon mass is of the order of (30) in any approximation. Thus, using (A.7) we obtain ultimately

$$m_s = \frac{\alpha}{V_0 R^2}\nu^{-5/12}\exp\left(\frac{2}{3\sqrt{3\nu}}\right), \tag{33}$$

where α is a number on the order of unity.

As shown earlier[10], the Curie temperature of a system of magnetic atoms of low density increases with ν like $T_C \sim \exp(-\nu^{1/2}\beta)$, $\beta \approx 1$. Consequently, the product $m_S T_C$, unlike in the high-density case, depends on ν. The Green's function at large q cannot be estimated by the method described above.

4. MAGNETIZATION

The average spin of the impurity is

$$\langle S_z\rangle = S - \int\frac{\rho(\omega)}{e^{\beta\omega}-1}d\omega, \tag{34}$$

where $\beta \approx T^{-1}$. At $\nu \gg 1$, substituting (20) in (34), we obtain

$$S - \langle S_z\rangle = \sqrt{\frac{2}{\pi}}\frac{\zeta(3/2)}{3\nu}\left(\frac{T}{\Theta}\right)^{3/2}, \tag{35}$$

where $\zeta(3/2)$ is the Riemann Zeta function

$$\Theta = 1/m_s R^2 \approx T_c. \tag{36}$$

We note the appearance of the parameter ν^{-1} in the right-hand side of (35). An analogous factor is contained in the expression for $S - \langle S_z\rangle$ in the papers of Vaks et al.[12], who investigated spin waves in a regular ferromagnet with a large exchange-interaction radius. In the case of low spin density, the deviation of $\langle S_z\rangle$ from saturation is again proportional to $T^{3/2}$, since the state density, as already noted, is proportional to $\sqrt{\omega}$. In this case, however, we cannot establish the concentration dependence of the coefficient at $T^{3/2}$, since we do not know the quantity $\operatorname{Im} L(q,\omega)$ at large q. We can only state that in the case $\nu \ll 1$ the real spin waves make a small contribution to $\rho(\omega)$, for in the opposite case it would turn out to be that $S - \langle S_z\rangle \sim c\nu^{-1}(T/\Theta)^{3/2}$, i.e., it depends on the volume of the unit cell. At low concentration of the randomly-distributed impurities, there should obviously be no such dependence.

5. ELECTRIC RESISTIVITY

Let us calculate the magnetic part of the electric resistivity of a ferromagnetic metal in which the magnetization is due to a system of randomly displaced localized spins, the exchange interaction of which is described by the Hamiltonian (1). An example of such a system, as already noted, is a diluted alloy of palladium with iron and cobalt. The indirect exchange interaction of localized spins is effected by the d electrons, and electric conductivity by the s electrons.

The Hamiltonian of the s-d exchange interaction of the electrons is

$$\begin{aligned}\mathscr{H}_{\mathrm{I}} = \frac{I}{N}\sum_l\sum_{k,k'}\varphi_l\{S_l^z(a_{k\uparrow}{}^+a_{k'\uparrow} - a_{k\downarrow}{}^+a_{k'\downarrow}) + S_l{}^+a_{k\uparrow}{}^+a_{k'\downarrow}\\ + S_l{}^-a_{k\downarrow}{}^+a_{k'\uparrow}\}\exp[i(\mathbf{k}-\mathbf{k}')r_l].\end{aligned} \tag{37}$$

The first term describes scattering of electrons without spin flip. If we replace in it S_l^z by $\langle S_l^z \rangle$, i.e., if we neglect the two-magnon scattering processes, then this term leads to elastic scattering by a randomly disposed system of impurities.

The temperature-dependent part $\Delta\rho_1(T)$ of the corresponding electric resistivity is due to the dependence of $\langle S_z \rangle$ on the temperature, and its value at $\nu \gg 1$ is

$$\Delta\rho_1(T) = -\sqrt{\frac{2}{\pi}}\,\zeta\left(\frac{3}{2}\right)\frac{m_e}{n_e e^2} SI^2 \frac{\Omega_0}{R^3} D_e(\varepsilon_F)\left(\frac{T}{\Theta}\right)^{3/2}, \tag{38}$$

where $D_e(\epsilon_F)$, m_e and n_e are respectively the density of states on the Fermi surface, and the effective mass and concentration of the s electrons. The remaining two terms of (37) describe inelastic single-magnon scattering of electrons. To calculate their contribution to the resistance, let us consider the collision term of a kinetic equation for the s electrons.

The probability of s-electron scattering by thermodynamic fluctuations of the transverse components of the localized spins is expressed in the usual manner in terms of the spectral density, which is connected with the imaginary part of the Green's function, so that the collision term can be written in the form:

$$\begin{gathered}\left(\frac{\partial f_{k\sigma}}{\partial t}\right)_{\text{c}\tau} = 4I^2 S\Omega_0 \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int d\omega\, \operatorname{Im} G(\omega, q)\left[N(\omega) + \frac{1}{2}\right. \\ \left. - \frac{\sigma}{2}\right] f_0(k,\sigma)\left[1 - f_0(k', -\sigma)\right]\left[g(k', -\sigma) - g(k,\sigma)\right] \\ \times \delta(\varepsilon_k - \varepsilon_{k'} + \sigma\omega)\,\delta(\mathbf{q} - \mathbf{k}' + \mathbf{k}),\end{gathered} \tag{39}$$

where σ characterizes the spin state of the s electron and takes on the values at ± 1

$$N(\omega) = (e^{\beta\omega} - 1)^{-1},$$

$f_0(k, \sigma)$ is the equilibrium electron distribution function, $g(k, \sigma)$ is a non-equilibrium increment defined by the relation

$$f(k,\sigma) = f_0(k,\sigma) + f_0(k,\sigma)\left[1 - f_0(k,\sigma)\right] g(k,\sigma).$$

According to (8), Im $G(q, \omega)$ consists of two terms. The first is connected with the Green's-function pole and corresponds to ordinary electron-magnon scattering. At $T \ll T_C$ this scattering is small-angle, and the corresponding contribution to the resistivity, as is well known, is proportional to T^2. The second term, which is connected with Im $L(q, \omega)$, does not depend on q at large $q \sim a^{-1}$ and $\omega \ll \omega_q$. It therefore leads to large-angle scattering of the electrons and it is possible to introduce into the collision term (39) a relaxation time τ, which is given for electrons on the Fermi surface, at $\nu \gg 1$, by

$$\frac{1}{\tau} = \frac{2^{3/2} - 1}{\pi}\,\zeta\left(\frac{3}{2}\right) I^2 S D_e(\varepsilon_F)\frac{\Omega_0}{R^3}\left(\frac{T}{\Theta}\right)^{3/2}, \tag{40}$$

i.e., the corresponding contribution to the resistivity is $\Delta\rho_2(T) \sim T^{3/2}$.

The total temperature-dependent part of the magnetic resistance is

$$\Delta\rho(T) = \Delta\rho_1(T) + \Delta\rho_2(T) = \frac{\sqrt{2} - 1}{\pi}\,\zeta\left(\frac{3}{2}\right)\frac{m_e}{n_e e^2} SI^2 D_e(\varepsilon_F)\frac{\Omega_0}{R^3}\left(\frac{T}{\Theta}\right)^{3/2}. \tag{41}$$

A similar dependence of $\Delta\rho$ on the temperature was obtained by Long and Turner[14] in a theoretical investigation of the electric resistivity of PbFe alloys. In the derivation of the Green's function of localized spins, Long and Turner neglected the fluctuations of the molecular field. In this sense, their model corresponds to the case of large spin density. However, the Green's function of the localized spins, which they obtained[14], does not contain $L(q, \omega)$ in the numerator, and therefore leads in fact to a quadratic dependence of the resistivity on the temperature. To obtain the $T^{3/2}$ dependence observed in experiment at $c < 1\%$, Long and Turner have arbitrarily left out the momentum conservation law from the collision integral (39). We note also that expression (41) for the resistivity differs from the corresponding expression in[14] by a factor c^{-1}.

In the case of low density of the magnetic atoms, $\Delta\rho(T)$ is also proportional to $T^{3/2}$. Inasmuch as the Green's function is known to us only at small q, we cannot establish the concentration dependence of the resistivity $\Delta\rho(T)$ connected with large-angle scattering.

It must be emphasized that at a sufficiently low impurity concentration, and at not too low temperatures, $\Delta\rho(T)$ can be determined by the electron scattering connected with local flipping of isolated spins[10].

6. CONCLUSION

In a system of randomly disposed spins, there exist weakly-damped long-wave magnons with a quadratic dispersion law. For a system with a large magnetic-atom intensity, we have obtained for the Green's function an expression that is valid at low frequencies for arbitrary values of the momentum. This has made it possible to derive in closed form experimentally-verifiable expressions for the thermodynamic and kinetic quantities.

In the case of low density of the magnetic atoms, we have a Green's function only for low momenta, near the pole. Since the density of states is determined by the short-wave part of $L(q, \omega)$, it follows that real spin waves connected with the pole $G(\omega, q)$ can be observed only in phenomena in which low frequencies and momenta are significant, say in small-angle neutron scattering. The corresponding measurements in alloys of the PdFe type, with low magnetic-atom concentration, would be quite desirable.

Williams and Lorem[15] determined the magnon mass in PdFe and PdCo alloys with Fe and Co concentrations lower than 1% ($\nu \ll 1$), using for the resistivity formulas of the type (41) obtained by Long and Turner[14]. It is clear from the foregoing that the concentration dependence obtained for the mass in this manner is not reliable.

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APPENDIX

We calculate, at $\nu \ll 1$, the quantity

$$\overline{H^{-1}} = \int_0^\infty \frac{W(H)}{H}\, dH. \tag{A.1}$$

The distribution function of the molecular field is given by[10]

$$W(H) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-D(\lambda) - i\lambda H}\, d\lambda, \tag{A.2}$$

$$D(\lambda) = -i\nu V_0 S\lambda \int_0^1 \ln^3 x \exp(-i\lambda V_0 S x)\, dx. \tag{A.3}$$

Substituting (A.2) in (A.1), we obtain

$$\overline{H^{-1}} = \frac{1}{2\pi}\int_0^\infty dH \int_{-\infty}^{\infty} d\left[\int_{-\infty}^{\lambda} e^{-D(t)}\, dt\right]\frac{e^{-i\lambda H}}{H}.$$

Integrating by parts and recognizing that[10]

$$\int_{-\infty}^{\infty} e^{-D(t)}\,dt = W(0) = 0,$$

we obtain

$$\overline{H^{-1}} = \frac{i}{2\pi}\int_{0}^{\infty} dH \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda H} \int_{-\infty}^{\lambda} e^{-D(t)}\,dt = i\int_{-\infty}^{0} e^{-D(t)}\,dt. \qquad (A.4)$$

Using expression (A.3) for $D(\lambda)$, we can show that $D(\lambda)$ is a regular function in the upper half-plane, and that at $|\lambda|SV_0 \gg 1$ we have the asymptotic form[10]

$$D(\lambda) = -\nu \ln^3(|\lambda|SV_0)\left[1 + O\left(\frac{1}{\ln|\lambda|SV_0}\right)\right]. \qquad (A.5)$$

Therefore, by deforming the contour, we can change over in (A.4) to integration along the imaginary semi-axis:

$$\overline{H^{-1}} = \int_{0}^{\infty} e^{-D(i\lambda)}\,d\lambda. \qquad (A.6)$$

It is seen from (A.2) that at $\lambda SV_0 \approx 1$ the argument of the exponential in (A.6) is $D(i\lambda) \approx \nu \ll 1$. Therefore the values important in (A.6) are $\lambda SV_0 \gg 1$, and the function $D(i\lambda)$ can be replaced by its asymptotic form (A.5). After simple transformations we then obtain

$$\overline{H^{-1}} = \frac{1}{V}\int_{0}^{\infty} \exp(-\nu t^3 + t)\,dt.$$

Integrating by the saddle-point method, we obtain ultimately

$$\overline{H^{-1}} = \frac{\pi^{1/2}}{V_0(3\nu)^{1/4}} \exp\left(\frac{2}{3\sqrt{3\nu}}\right) \qquad (A.7)$$

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[1] We note that the inequality $\nu \gg 1$ is the condition for applicability of the molecular-field method near the Curie temperature [12]. In the considered system, the Curie temperature T_c, obtained by the molecular-field method, is proportional, as is also m_s^{-1}, to the concentration c.

[1] T. Kaneyoshi, Progr. Theor. Phys., 42, 477 (1969)

[2] S. F. Edwards and R. C. Jones, J. Phys., C4, 2109 (1971).

[3] R. A. Tahir-Kheli, Phys. Rev., B6, 2808 (1972).

[4] J. Crangle and W. R. Scott, J. Appl. Phys., 36, 921 (1965).

[5] R. M. Bozorth, P. A. Wolff, P. P. Davis, V. B. Compton, and J. H. Wernik, Phys. Rev., 122, 1757 (1961).

[6] G. Williams and J. W. Loram, J. Phys. Chem. Sol., 30, 1827 (1969). G. Williams, J. Phys. Chem. Sol., 31, 529 (1970).

[7] C. J. Shinkel, F. R. DeBoer, and J. Bisterbos, Phys. Lett., 26A, 601 (1968).

[8] A. T. Aldred, B. D. Rainfold, and M. W. Stringfellow, Phys. Rev. Lett., 24, 897 (1970).

[9] G. G. Low and J. M. Holden, Proc. Phys. Soc., 89, 119 (1966).

[10] I. Ya. Korenblit and E. F. Shender, Zh. Eksp. Teor. Fiz. 62, 1949 (1972) [Sov. Phys.-JETP 35, 1017 (1972)].

[11] T. Matsubara and F. Yonezawa, Progr. Theor. Phys., 34, 871 (1965).

[12] V. G. Vaks, A. I. Larkin, and S. A. Pikin, Zh. Eksp. Teor. Fiz. 53, 281, 1089 (1967) [Sov. Phys.-JETP 26, 188 (1968)].

[13] J. Callaway, Phys. Rev., 132, 2003 (1963).

[14] P. D. Long and R. E. Turner, J. Phys. Chem., Metal Phys., Suppl., N2, S127 (1970).

[15] G. Williams and J. W. Loram, J. Phys.,Fluids, Metal. Phys., 1, 434 (1971).

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