Free electrons on the surface of liquid helium in the presence of external fields

V. B. Shikin and Yu. P. Monarkha

Physico-technical Institute of Low Temperatures, Ukrainian Academy of Sciences

(Submitted March 9, 1973)
Zh. Eksp. Teor. Fiz. **65**, 751–761 (August 1973)

A detailed study is made of the localization of surface electrons at the liquid-vapor interface of liquid helium in a strong pressing electric field. The concept of a surface anion is introduced. Its effective volume, mass, and coupling energy are calculated. The mobility of surface anions for large and small Reynolds numbers are calculated in the hydrodynamic limit.

External electric and magnetic fields exert a noticeable influence on the structure of surface electronic states in liquid helium. This question was discussed in part by one of the authors[1]. However, many interesting possibilities have not yet been cleared up. We have in mind, for example, the localization of surface electrons in a strong "clamping" electric field $E_\perp$, galvanomagnetic phenomena in which surface electrons take part, etc. The present paper contains a detailed discussion of these questions.

SURFACE ELECTRONS IN A STRONG FIELD $E_\perp$

1. A free electron located in the gas near the liquid-vapor boundary is attracted to this boundary by the known electrostatic-image forces. The field of the image forces ensures localization of the electron over the free surface of the helium at a distance z_0[1] (the z axis is normal to the interface, the gas phase corresponds to $z \le 0$):

$$z_0 = \frac{4\hbar\varepsilon(\varepsilon+1)}{me^2(\varepsilon-1)}, \tag{1}$$

where m and e are the mass and charge of the electron, and ϵ is the dielectric constant of the liquid helium. (For gas we have $\epsilon \approx 1$.) Putting $\epsilon - 1 = 0.06$ we get $z_0 \approx 10^{-6}$ cm.

We now turn on a field $E_\perp$ of intensity such that the localization of the electron over the surface of the helium is determined mainly by $E_\perp$. Neglecting in this limit the image forces, we can, as in the calculation of (1), obtain an exact solution of the Schrödinger equation for the electron wave function ψ satisfying the boundary conditions

$$\psi|_{z\to-\infty} \to 0, \quad \psi|_{z=0} = 0.$$

The function $\psi(x, y, z)$ is given by

$$\psi(x, y, z) = \mathrm{const}\, f_n(z) \exp[i(k_x x + k_y y)],$$
$$f_n(z) = F\left[\left(\frac{2m\grave{e}E_\perp}{\hbar^2}\right)^{1/3}\left(-z - \frac{\lambda_\perp^{(n)}}{eE_\perp}\right)\right], \quad z \leqslant 0, \tag{2}$$
$$\lambda_\perp^{(n\gamma} = \zeta_n E_\perp \left(\frac{\hbar^2}{2meE_\perp}\right)^{1/3}, \quad n = 1, 2, 3, \ldots,$$

where F(z) is an Airy function and ζ_n are the zeroes of this function ($\zeta_1 = 2.34$).

In the ground state (2) the electron is localized over the surface at a distance $z_1 \approx 1.3(\hbar^2/2meE_\perp)^{1/3}$. The criterion for the validity of the approximation (2) is the requirement $z_1 \lesssim z_0$, where z_0 is given by (1). In expanded form, this inequality serves as a definition of the demarcation field $E_\perp^0$ between the weak fields $E_\perp < E_\perp^0$ and strong fields $E_\perp > E_\perp^0$

$$E_\perp{}^0 \approx \frac{\hbar^2}{2me}\left(\frac{1.3}{z_0}\right)^3. \tag{3}$$

The numerical value of $E_\perp^0$ is approximately 5–10 cgs esu.

It should be noted that in fields $E_\perp \gg E_\perp^0$ the electronic surface states (2) are metastable, since there is a finite probability of electron tunneling into the liquid phase. In fact, the potential barrier for an electron in the field $E_\perp$, neglecting the image forces, takes the form of a sawtooth step:

$$U(z) = \begin{cases} -eE_\perp z, & z < 0, \\ U_0 - eE_\perp z, & z > 0, \end{cases} \tag{4}$$

where U_0 is the energy of penetration of the electron into the liquid helium ($U_0 \approx 1$ eV). The probability w of tunneling through such a barrier can be estimated with the aid of relations taken from the theory of cold emission[2]

$$w = \mathrm{const}\exp\left[-\frac{8\pi(U_0 - \lambda_\perp{}^1)^{3/2}\sqrt{2m}}{3e\hbar E_\perp}\right]. \tag{5}$$

The level $\lambda_\perp^1$ corresponds to the energy of the ground state of the electron from (2). On the basis of (5) and (2) we can conclude that for fields $E_\perp \lesssim 10^2$ cgs esu the probability of tunneling of an electron into the liquid phase is negligibly small. The effect becomes noticeable at $E_\perp \sim 10^3 - 10^4$ cgs esu. On the other hand, breakdown phenomena occur in strong fields[3] and impose an upper bound on the numerical value of the parameter $E_\perp$.

2. We proceed now to refine the solution (2), by taking into account the deformation of the free surface of the helium in the region where the electron is localized, and the influence of this deformation on the structure of the electron wave function. As noted first in[1], the appearance of a deformation of the surface of helium leads to a localization of the electron also in the plane of the free surface. The system of equations describing the structure of the resultant complex is[1]

$$\begin{gathered} \psi(r, z) \approx f_n(z)\varphi(r); \\ \frac{\hbar^2}{2m}\Delta\varphi + (eE_\perp\xi + \lambda_\parallel)\varphi = 0, \\ \alpha\Delta\xi - \rho g\xi = p_{\mathrm{el}}(r), \quad p_{\mathrm{el}}(r) = eE_\perp\varphi^2(r); \\ \varphi'(0) = \varphi(\infty) = 0, \quad \xi'(0) = \xi(\infty) = 0, \\ 2\pi\int_0^\infty \varphi^2(r)\, r\, dr = 1, \end{gathered} \tag{6}$$

where $\varphi(r)$ is the wave function of the electron and describes its localization along the surface, r and Δ are the two-dimensional radius vector and Laplacian, $\xi(r)$ is the local deviation of the shape of the surface from the equilibrium flat state, α and ρ are the coefficients of surface tension and the bulk density of the liquid helium, g is the acceleration due to gravity, and $\lambda_\parallel$ is the energy of electron localization in the r direction.

It is convenient to begin the solution of (6) with a discussion of several properties of the Schrödinger equation for the electron. To this end we assume that the self-consistent localization of the electron in the field $eE_\perp\xi(r)$, in which we are interested, does indeed take place, and that the scale of localization of the electron is the characteristic length L. In addition we assume that in the region of electron localization the coordinate dependence of $\xi(r)$ is likewise determined only by the scale L. Neglecting on this basis the gravitational term $\rho g\xi(r)$ in the equation for $\xi(r)$ in the region $0 \leq r \ll \kappa^{-1}$, we can represent $\xi(r)$ in the form

$$\xi(r)=\xi(0)+\frac{eE_\perp}{\alpha}\int_0^r \frac{ds}{s}\int_0^s \varphi^2(x)x\,dx, \qquad 0\leqslant r\ll\varkappa^{-1}, \quad \varkappa^2=\rho g/\alpha. \tag{7}$$

Using (7), we obtain now an explicit expression for the scale L. Introducing for this purpose the substitution

$$\varphi(r)=C\phi(x), \quad x=r/L, \tag{8}$$

where C is an undetermined constant, substituting (7) and (8) in equation (6) for the electron, and subjecting the constants C and L to the requirements

$$C^2L^2=\frac{1}{8\pi}, \qquad \frac{\hbar^2}{2mL^2}=\frac{1}{\alpha}(eE_\perp)^2L^2C^2, \tag{9}$$

we obtain for $\phi(x)$

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{\partial\phi}{dx}\right)+\left[\int_0^x\frac{ds}{s}\int_0^s\phi^2(t)t\,dt+\bar{\lambda}_\parallel\right]\phi=0;$$

$$\phi'(0)=\phi(\infty)=0, \qquad 4\int_0^\infty\phi^2(x)x\,dx=1, \tag{10}$$

$$\bar{\lambda}_\parallel=\frac{2mL^2}{\hbar^2}[eE_\perp\xi(0)+\lambda_\parallel].$$

The absence of dimensional factors in (7) signifies that the only length scale in the electronic problem is the quantity L from (9):

$$L^2=4\pi\alpha\hbar^2/m(eE_\perp)^2.$$

The condition for the self-consistency of the definition (7) of $\xi(r)$, which enables us to match (7) to the correct solution for $\xi(r)$ at infinity, is the requirement $\kappa L \ll 1$. Using the definitions of L and κ, we can easily verify that this inequality does indeed hold for all $E_\perp > E_\perp^0$. Thus, if $E_\perp \gtrsim 10$ cgs esu, recognizing that $\alpha = 0.36$ erg/cm^2 and $\rho = 0.146$ g/cm^3, we obtain $L \sim 10^{-5}$ cm, $\kappa^{-1} \sim 10^{-1}$ cm, and $\kappa L \approx 10^{-4} \ll 1$.

It should be noted that the numerical gauge $C^2L^2 = 1/8\pi$ in (9) is valid only accurate to a factor ~1. Its concrete choice is governed by the following considerations. If we expand the resultant deformation $\xi(r)$ about the origin in powers of r.

$$\xi(r)=\xi(0)+{}^1\!/_2\xi''(0)r^2+\dots,$$

then it can be easily shown, if we have the complete solution of the problem, that the expansion is in fact in powers of r/L. This means that when determining $\varphi(r)$ or $\phi(x)$ we can replace $\xi(r)$ in the region $0 < r \lesssim L$ by the indicated expansion and terminate this expansion with the quadratic term.

As a result, the wave equation assumes an oscillatory form and all the characteristics of the ground state of the localized electron can be expressed explicitly:

$$\varphi^2(r)=\frac{2}{\pi L^2}\exp\left(-\frac{2r^2}{L^2}\right); \qquad \lambda_\parallel{}^0=-eE_\perp\xi(0)+\bar{\lambda}_\parallel{}^0\hbar^2/2mL^2, \quad \bar{\lambda}_\parallel{}^0=4, \tag{11}$$

where L coincides exactly with (9). The last circumstance was indeed the formal reason for choosing the gauge $C^2L^2 = 1/8\pi$ in (9).

Having determined the characteristic scale of the wave function $\varphi(r)$, we now obtain the distribution of the displacement $\xi(r)$ of the free surface of helium under the influence of the electron pressure $p_{el}(r)$. Using the Bessel transformation of the equation of mechanical equilibrium under the assumption that the perturbing pressure $p_{el}(r)$ has cylindrical symmetry and acts on a finite surface area, i.e., $p_{el} \to 0$ as $r \to \infty$, we obtain for the displacements $\xi(r)$ the expression

$$\xi(r)=\frac{1}{\alpha}\int_0^\infty G(\omega)J_0(\omega r)\omega\,d\omega,$$

$$\xi'(r)=-\frac{1}{\alpha}\int_0^\infty G(\omega)J_1(\omega r)\omega^2\,d\omega; \tag{12}$$

$$G(\omega)=\frac{P(\omega)}{\omega^2+\varkappa^2}, \qquad P(\omega)=\int_0^\infty p_{el}(r)J_0(\omega r)r\,dr,$$

$J_0(x)$ and $J_1(x)$ are Bessel functions.

At an electron pressure $p_{el}(r) = eE_\perp\varphi^2(r)$, with the wave function $\varphi(r)$ chosen in the form (11), we obtain

$$G(\omega)=\frac{eE_\perp}{2\pi(\omega^2+\varkappa^2)}\exp\left(-\frac{L^2\omega^2}{8}\right). \tag{12a}$$

The asymptotic forms (12) for small and large distances then become

$$\xi(0)\approx\frac{eE_\perp}{4\pi\alpha}\ln\frac{8}{\gamma\varkappa^2L^2}, \qquad \xi(r)|_{r\to\infty}\approx\frac{eE_\perp}{2\pi\alpha}K_0(\varkappa r) \tag{13}$$

($\gamma = 1.78$ is the Euler number and $K_0(x)$ is a Macdonald function). It follows from (13) that the radius of the deformation of the free surface under the influence of p_{el}, as expected, is of the order of κ^{-1}. At $E_\perp \gtrsim 10$ cgs esu, $\alpha = 0.36$ erg/cm^2, and $\rho = 0.146$ g/cm^3 we have $\xi(0) \gtrsim 10^{-8}$ cm.

The volume of the surface anion, defined by the relation

$$v_s=2\pi\int_0^\infty\xi(r)r\,dr=\frac{2\pi}{\alpha\varkappa^2}\int_0^\infty p_{el}(r)r\,dr=\frac{eE_\perp}{\alpha\varkappa^2}, \tag{14}$$

assumes tremendous values (in comparison with the anion volume v_0 in the interior of the liquid, $v_0 = 4\pi R_0^3/3 \approx 3 \times 10^{-20}$ cm^3). Thus, at $E_\perp \approx 10$ cgs esu we have $v_s = 10^{-10}$ cm^3 and $v_s/v_0 \sim 10^9$–10^{10}. It is also of interest to compare the masses associated with these formations. A concrete comparison of the masses will be made later on, after calculating the field of the velocities produced around the surface anion moving along the liquid-vapor interface. We note here only that it does not reduce to a ratio of the volumes.

Expression (11) for the localization energy, with allowance for (13), takes the form

$$\lambda_\parallel{}^0 = -\frac{(eE_\perp)^2}{4\pi\alpha}\left[\ln\frac{8}{\gamma\varkappa^2L^2}-2\right]. \qquad (11a)$$

At $\kappa L \ll 1$, the quantity $\ln(8/\gamma\kappa^2L^2) \gg 2$, and the requirement $\lambda_\parallel^0 < 0$ is satisfied. The numerical value of $\lambda_\parallel^0$ at $E_\perp \gtrsim 10$ cgs esu turns out to be $\lambda_\parallel^0 \gtrsim 1^\circ K$. This is sufficient localization energy for a stable existence of surface anions at temperatures $\lesssim 0.1^\circ K$. In fact, let us calculate the total energy connected with the formation of a surface anion:

$$\mathscr{E} = \int_S \left[\frac{\alpha}{2}(\nabla\xi)^2 + \frac{1}{\alpha}\rho g\xi^2 + \frac{\hbar^2}{2m}(\nabla\varphi)^2 - eE_\perp\xi\varphi^2\right] dS.$$

Calculation of this integral with allowance for the explicit forms of $\xi(r)$ and $\varphi(r)$ yields

$$\mathscr{E} = -\frac{(eE_\perp)^2}{8\pi\alpha}\left[\ln\frac{4}{\gamma\varkappa^2L^2}-2\right],$$

i.e., $\mathscr{E}$ differs from $\lambda_\parallel^0$ of (11a) only by a numerical factor ~1.

3. The wave function of the electron can be localized in the plane of the free surface of the liquid also by other means which are not connected with the deformation of the surface $\xi(r)$. Thus, surface anions are always produced in the presence of a magnetic field perpendicular to the interface. The anion localization radius coincides in this case with the radius of the ground state of the electron in the magnetic field. Taking into account the exponential form of the wave function of the electron in this state, we can readily conclude that the deformation of the surface under the electron will have the same form (12) as before, if the length L is taken to mean $L_H = 4c\hbar/EH$, where e is the electron charge, c is the velocity of light, and H is the intensity of the external magnetic field.

We consider further the following situation: let a volume anion introduce into the He^3—He^4 solution be near the lamination boundary on the side enriched with He^3. In this case, the electrostatic image forces attract the anion towards the lamination boundary. For the anion, on the other hand, this boundary itself is a potential barrier of height ~100°K because the surface tension on the anion surface depends on the He^3 concentration (the height of the barrier was calculated earlier in[4]). As a result, the anion is localized on the lamination boundary with a very small localization radius, on the order of the volume anion radius $R_0 \sim 20$ Å. When $E_\perp$, which presses the ion towards the lamination boundary, is turned on, the result is a deformation of the lamination boundary, qualitatively analogous to (12). For numerical estimates we can use in which case the mechanical equilibrium equation (6), in which we put

$$p_{el}(r) = \begin{cases} eE_\perp/\pi R_0^2, & r \leqslant R_0 \\ 0, & r > R_0 \end{cases}.$$

As a result, the distribution of the displacements takes the form (12) with

$$G(\omega) = \frac{eE_\perp}{\pi\omega R_0(\omega^2+\varkappa^2)} J_1(\omega R_0). \qquad (15)$$

In addition, the coefficient α should be replaced by α^*, the coefficient of surface tension on the lamination boundary.

An analogous deformation is produced on the vapor-liquid interface, if the electron is brought to the surface from the side of the gas phase, and the gas density is already high enough for bubbles to be produced in the gas, i.e., $n \gtrsim n_{cr} \approx 10^{21}$ cm^{-3}[5]. It is appropriate to note here also the following possibility. Assume that the gas density is $n \lesssim n_{cr}$. In this case, localization of the electrons is energywise inconvenient in the volume of the gas phase, primarily because in three-dimensional problems bound levels arise only in a well of finite depth. As to an electron brought close to a liquid-vapor boundary, its wave function is localized in a direction perpendicular to the surface, either by the image forces or by the external $E_\perp$. As a result, the problem of electron localization in a gas becomes two-dimensional and consequently the localization conditions become much easier. The resultant situation is perfectly analogous to that described in a paper by Kukushkin and one of the authors[5], the only difference being that in[5] the decrease in the dimensionality of the problem of electron localization in gaseous helium was attained by making use of a strong magnetic field. Estimates of the localization energy λ_{gas} of a surface electron in gaseous helium lead to the following relations:

$$\lambda_{gas} = T\left(1-\frac{n_c}{n}\right)\frac{z_1}{a_0}, \qquad n_c = \frac{4T}{3\pi}\frac{mz_1}{\hbar^2a_0^2} \qquad (16)$$

(a_0 is the scattering length of an electron by an individual helium atom, $a_0 = 0.62$ Å, z_1 is obtained from (2), and T is the temperature). It is seen from the definition of λ_{gas} that such states arise only when $n > n_c$. In the region $n > n_{cr} \approx 10^{21}$ cm^{-3}, on the other hand, volume localization begins. Consequently, for the states (16) to exist it is necessary to satisfy the inequality $n_c < n_{cr}$. Such an inequality does indeed begin to be satisfied at $T \sim 1$–$3^\circ K$ for $z_1 \lesssim 10^{-6}$ cm, which corresponds to fields $E_\perp \gtrsim 10^1$–10^2 cgs esu.

4. It is of interest to trace the tendency to localization of three surface electrons with small wave numbers in a strong "clamping" field $E_\perp$. To this end, we calculate the correction $eE_\perp\xi(r)$ to the dispersion law of a free electron as a result of its interaction with capillary waves. In second-order perturbation theory, taking into account the concrete form of the resultant matrix elements, which are analogous to those used earlier[1], we have

$$\varepsilon(\mathbf{k}) - \varepsilon_0(\mathbf{k}) = \sum_{\mathbf{q}} \frac{w_q{}^2}{\varepsilon_0(\mathbf{k}) - \varepsilon_0(\mathbf{k}+\mathbf{q}) - \hbar\omega_q};$$

$$\varepsilon_0(\mathbf{k}) = \frac{\hbar^2k^2}{2m}, \qquad w_q{}^2 = \frac{e^2E_\perp{}^2}{S}\frac{\hbar q}{2\rho\omega_q},$$

where m is the mass of the free electron, k and q are the wave numbers of the electron and surface phonons, S is the free-surface area, and $\omega_q^2 = \alpha q^3/\rho$. Changing over from summation to integration

$$\sum_{\mathbf{q}} \to \frac{S}{(2\pi)^2}\int d\mathbf{q}$$

and integrating with respect to the angle variable, we reduce the expression for the dispersion law to the form

$$\Delta\varepsilon_k = \gamma\frac{e^2E_\perp{}^2m}{2\pi\sqrt{2\alpha\rho}\,\hbar k^{1/2}}, \qquad \gamma = \int_0^1 \frac{ds}{s^{1/2}(1-s^2)^{1/2}} \approx 1. \qquad (17)$$

It follows from (17) that as $k \to 0$ the change $\Delta\epsilon_k$ of the dispersion law is no longer small, a fact corresponding to a tendency to localization. The region of applicability of (17) is defined by the inequality $\Delta\epsilon_k/\epsilon_0(\mathbf{k}) \ll 1$. At $E_\perp \sim 1$ cgs esu, this inequality is satisfied for $k \gtrsim 10^4$ cm^{-1}. Thus, at $k \sim 10^5$ cm^{-1}, we have $\Delta\epsilon_k/\epsilon_0(\mathbf{k}) \approx 10^{-3}$.

DYNAMIC PROPERTIES OF SURFACE ANIONS

1. We define a surface anion as an electron combined with the surface deformation resulting under the electron in the "clamping" field, and assume that the surface anion moves under the influence of the field $E_\perp$ with uniform velocity along the substrates. The local deformation of the helium surface under the electron, which we shall assume to have time to adjust itself adiabatically to the moving electron, should induce, as it moves, a definite field of hydrodynamic velocities. The perturbation of the liquid can indeed be described here in hydrodynamic terms, since the characteristic scale of the perturbation L is assumed to be large in comparison with the mean free path L of the thermal excitations in liquid helium: $L \gg l$.

It is convenient to solve the problem of finding the velocity field in a coordinate system r, θ, z that moves together with the center of the anion (the liquid phase corresponds to $z \geq 0$). In this coordinate system, the surface deformation $\xi(r)$ is cylindrically symmetrical and immobile, while the liquid moves with velocity V_0 at infinity. The total velocity

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}$$

($\mathbf{v}$ is the velocity field induced by the surface anion) should obviously satisfy the boundary condition

$$\mathbf{Vn} = 0$$

(n is the normal to the deformed surface). Taking into account the explicit form of the direction cosines

$$n_r = \frac{\partial \xi}{\partial r} = \xi'(r) \ll 1, \qquad n_\theta = 0, \qquad n_z = -1;$$

this condition can be rewritten as

$$v_z|_{z=0} = V_0 \cos\theta \xi'(r), \tag{18}$$

where the angle θ is reckoned from the direction of the electric field $E_\parallel$ and $\xi'(r)$ is given by (12).

Proceeding to calculate the field v, we consider first the limit of large Reynolds numbers, while the viscous terms in the equations of motion can be neglected. The velocity v is then determined from the equations

$$\begin{gathered} \mathbf{v} = \nabla\chi, \quad \Delta\chi = 0; \\ \chi|_{r,z\to\infty} \to 0, \qquad \left.\frac{\partial \chi}{\partial z}\right|_{z=0} = V_0 \xi'(r) \cos\theta. \end{gathered} \tag{19}$$

The solution of (19) takes the form

$$\chi(\mathbf{r}, z) = V_0 \frac{\cos\theta}{\alpha} \int_0^\infty G(\omega) e^{-\omega z} J_1(\omega r) \omega \, d\omega.$$

Accordingly

$$\begin{gathered} v_r = V_0 \frac{\cos\theta}{\alpha} \int_0^\infty G(\omega) e^{-\omega z} J_1'(\omega r) \omega^2 \, d\omega, \\ v_\theta = -V_0 \frac{\sin\theta}{\alpha r} \int_0^\infty G(\omega) e^{-\omega z} J_1(\omega r) \omega \, d\omega, \\ v_z = -V_0 \frac{\cos\theta}{\alpha} \int_0^\infty G(\omega) e^{-\omega z} J_1(\omega r) \omega^2 \, d\omega. \end{gathered} \tag{20}$$

The quantity $G(\omega)$ is given either by (12a) or (15). The asymptotic components of the velocity at large distances from the surface anion can be easily expressed in analogy with (13), but there is no particular need for this.

After determining the velocity field (20), we calculate the mass associated with the surface anion

$$M = -\frac{\rho}{V_0^2} \int_S \chi \frac{\sigma\chi}{\partial n} dS = \frac{\pi\rho}{\alpha^2} \int_0^\infty G^2(\omega) \omega^2 \, d\omega = \frac{\rho (eE_\perp)^2}{16\alpha^2 \varkappa}. \tag{21}$$

It is seen from (21) and (14) that there is no direct connection between the volume of the anion and its associated mass in the surface case (unlike the volume anion, where the associated mass M_0 is proportional to the volume of the anion). Comparing the mass (21) with the associated mass M_0

$$M_0 = \frac{2}{3} \pi R_0^3 \rho, \qquad R_0 \approx 20 \text{Å}, \qquad \frac{M}{M_0} = \frac{3 (eE_\perp)^2}{32 \alpha^2 \varkappa \pi R_0^3},$$

we find that at $E_\perp > E_\perp^0$ we have $M/M_0 \gg 1$. For example, for $E_\perp \approx 10$ cgs esu we have $M/M_0 \approx 10^2$.

2. The next important dynamic characteristic of the surface anion is its mobility along the surface. In the case of a normal liquid it is convenient to define the mobility, or equivalently the average velocity V_0 under the influence of the field $E_\parallel$, by using the energy balance equation

$$-eE_\parallel V_0 = W = \eta \int_S \frac{\partial v^2}{\partial n} dS, \tag{22}$$

where η is the first viscosity coefficient. The concrete expression used in (22) for the total dissociation energy W holds when $\mathbf{v} = \nabla\chi$ and $\nabla^2\chi = 0$ (see, for example,[6], page 73). On the basis of (22) and (20) we have for the mobility $\mu = V_0/E_\parallel$ the following general expression:

$$\mu = \alpha^2 e / 4\pi\eta \int_0^\infty G^2(\omega) \omega^4 \, d\omega. \tag{23}$$

In the case of a concrete distribution of the pressure (see (11) and (12a)) it follows from (23) that

$$\mu = \frac{\sqrt{\pi} \alpha^2 L}{\eta e E_\perp^2} = \frac{2\pi\hbar\alpha^{5/2}}{\eta m^{1/2} e^2 E_\perp^3}. \tag{23a}$$

It is convenient to compare this value of the mobility with that of a hard sphere of radius R_0 in a viscous liquid:

$$\frac{\mu}{\mu_0} = \frac{4\pi^{3/2} R_0 L}{e^2 E_\perp^2}. \tag{24}$$

According to (24), in fields $E_\perp > 10$ cgs esu the surface anions have a mobility larger by two or three orders of magnitude than a sphere of radius $R_0 \approx 20$ Å. It is appropriate to note that, unlike the associated mass and the volume, the mobility of the surface anions does not depend on the parameter $\kappa L \ll 1$. The gravitational term has been introduced into the mechanical-equilibrium equation (6) to ensure good convergence of all the results at large distances from the center of the anion. In the classical situation, when the deformation $\xi(r)$ is macroscopic in scale (in our case this can be attained by clamping the electron to the surface with a field $E_\perp \sim 10^2$–10^3 cgs esu), there is no doubt that it is reasonable to introduce the gravitational cutoff at distances $\sim\kappa^{-1}$. On the other hand, if $\xi(0) \lesssim 10^{-8}$ cm, then the fluctuation displacements of the surface become comparable with the scale of the self-consistent deformation, and the retention of the scale κ^{-1} in the final results for the volume and mass of the surface anion calls apparently for a more consistent justification than given above. As to the mobility (see (24)), in spite of the classical derivation of this relation, the final result does not contain κ, and can be regarded as reasonable in the field region $E_\perp \lesssim 10^{-7}$ cm.

3. To calculate the mobility in the limit of small

Reynolds numbers it is necessary to solve the following system of equations:

$$\nabla p = \eta \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{v} = 0 \tag{25}$$

with boundary conditions (18) and

$$\sigma_{zr}|_{z=0} = 0, \quad \sigma_{z\theta}|_{z=\theta} = 0 \tag{25a}$$

Here p is the pressure in the liquid and σ_{zi} are the tangential components of the viscous-stress tensor.

A general solution of (25), which vanishes at infinity, is

$$\begin{gathered}
p = \eta \cos\theta \int_0^\infty \Phi(\omega) e^{-\omega z} J_1(\omega r)\, \omega\, d\omega; \\
v_r = \frac{1}{4} \cos\theta \int_0^\infty \Phi(\omega) e^{-\omega z} \{[a(\omega) + \omega z] J_2(\omega r) - [b(\omega) + \omega z] J_0(\omega r)\}\, d\omega, \\
v_\theta = \frac{1}{4} \sin\theta \int_0^\infty \Phi(\omega) e^{-\omega z} \{[a(\omega) + \omega z] J_2(\omega r) + [b(\omega) + \omega z] J_0(\omega r)\}\, d\omega, \\
v_z = \frac{1}{4} \cos\theta \int_0^\infty \Phi(\omega) e^{-\omega z} [a(\omega) + b(\omega) + 2 + 2\omega z] J_1(\omega r)\, d\omega.
\end{gathered} \tag{26}$$

The arbitrary functions $a(\omega)$, $b(\omega)$, and $\Phi(\omega)$ are obtained from the boundary conditions. Substituting (26) in (25a) and (18), we obtain

$$a(\omega) = b(\omega) = 0, \qquad \Phi(\omega) = -\frac{2V_0}{\alpha} \omega^2 G(\omega). \tag{26a}$$

The energy-balance equation with allowance for the concrete behavior of the velocity components (26)—(26a) at z = 0, namely

$$v_\theta = v_r = \partial v_z / \partial z = 0$$

is expressed in this case in the form

$$eE_\parallel V_0 = \eta \int (\operatorname{rot} \mathbf{v})^2\, d\mathbf{r}\, dz. \tag{27}$$

Calculating the required components

$$\begin{gathered}
(\operatorname{rot} \mathbf{v})_r = \frac{2V_0}{\alpha r} \sin\theta \int_0^\infty G(\omega) e^{-\omega z} J_1(\omega r)\, \omega^2\, d\omega, \\
(\operatorname{rot} \mathbf{v})_\theta = \frac{2V_0}{\alpha} \cos\theta \int_0^\infty G(\omega) e^{-\omega z} J_1'(\omega r)\, \omega^3\, d\omega, \\
(\operatorname{rot} \mathbf{v})_z = 0
\end{gathered}$$

and integrating in (27), we obtain ultimately the following result for the mobility μ:

$$\mu = \alpha^2 e / 2\pi\eta \int_0^\infty G^2(\omega)\, \omega^4\, d\omega. \tag{28}$$

This expression differs from (23) only by a factor of 2. A similar situation arises also at different variants of the calculation of the Stokes force for a spherical bubble. Different limiting cases with respect to the Reynolds number alter in the Stokes force only the numerical coefficients, and leave the symbolic expression unchanged.

CONCLUSION

Let us list the summary results. As noted earlier[1] and as investigated in detail in[7], electrons that are localized near the liquid-vapor boundary by electrostatic image forces only interact weakly with the oscillations of the free surface of the helium. This conclusion is confirmed qualitatively by the experimental data on the mobility of the electrons along the interface. In all the known attempts to determine this mobility[8-11], there was either no clamping field at all, or else it was small enough ($E_\perp \lesssim 100$ V/cm). As a result, the observed mobility practically coincided with the mobility of the electrons in the volume of the gaseous phase of helium, thus indicating no noticeable contribution to the total electron mobility by electron scattering from the surface oscillations.

In this situation, the surface electrons can be actually used to study the dynamic properties of the surface of liquid helium only apparently by resorting to a strong field $E_\perp$. In this case the electron mobility along the surface has a clear-cut parametric dependence on $E_\perp$, and this dependence is different for the case of surface anions, where $\mu \propto E_\perp^{-3}$ (see (23a) and (28)) from the case of free surface electrons, for which, as follows from[1]

$$\mu = 8\hbar\alpha / emE_\perp^2;$$

thus, $\mu \propto E_\perp^{-2}$. In addition, the mobility μ in the field $E_\perp$ differs qualitatively from μ_{gas} in the sense of the temperature dependence, and competes in magnitude with μ_{gas} at $E_\perp \gtrsim 10$ cgs esu even in the region $T \sim 1^\circ$K, let alone lower values of T, where μ_{gas} can be neglected with exponential accuracy.

It should be noted that the foregoing calculation of the mobility of the surface anions does not take into account the possible absorption phenomena on the liquid-vapor boundary of the superfluid helium. In this connection, the results (23) and (28) are valid to full degree only for liquid He^3. As to the influence of the "normal" surface component on the mobility of the surface anions, it calls for a special analysis.

The authors are grateful to A. F. Andreev for a critical discussion of the results.

---

[1] V. B. Shikin, Zh. Eksp. Teor. Fiz. 60, 713 (1971) [Sov. Phys.-JETP 33, 387 (1971)].

[2] E. V. Shpol'skiĭ, Atomnaya fizika (Atomic Physics), Vol. 1, Fizmatgiz, 1963.

[3] C. Blank and M. H. Edwards, Phys. Rev., 119, 50 (1960).

[4] V. B. Shikin, Zh. Eksp. Teor. Fiz. 64, 1414 (1973) [Sov. Phys.-JETP 37, No. 4 (1973)].

[5] J. L. Levin and T. M. Sanders, Phys. Rev. 154, 138 (1967); L. S. Kukushkin and V. B. Shikin, Zh. Eksp. Teor. Fiz. 63, 1830 (1972) [Sov. Phys.-JETP 36, 969 (1973)].

[6] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953 [Pergamon, 1958].

[7] V. B. Shikin and Yu. P. Monarkha, in: Fizika kondensirovannogo sostoyaniya (Physics of the Condensed State), Trudy FTINT AN USSR No. 18, (1972), p. 75.

[8] V. B. Shikin, Yu. Z. Kovdrya, and A. S. Rybalko, ibid., No. 15 (1971), p. 99.

[9] W. T. Sommer and D. J. Tanner, Phys. Rev. Lett., 27, 1345 (1971).

[10] R. M. Ostermeier and K. W. Schwarz, Phys. Rev. Lett., 29, 25 (1972).

[11] T. R. Brown and C. G. Grimes, Phys. Rev. Lett., 29, 1233 (1972).

Translated by J. G. Adashko
78